Equilibrium conditions and symmetries for foams in contact with solid surfaces

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Abstract

The patterns formed by liquid foams in contact with solid surfaces resemble 2D foams. We derive the equilibrium equations for such 2D foams when the solid surface is curved and smooth, generalising the standard case for flat Hele–Shaw cells. The equilibrium conditions at the vertices in 2D are invariant under conformal transformations. We establish the transformation rules for films under conformal maps. Even if conformal invariance does not hold in general, by considering foams confined between two closely spaced non-parallel plates, we show that an exponential profile gives a pattern that resembles the image of a regular hexagonal foam under a complex logarithm map, to lowest order.

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1. Introduction

Several recent experiments showed images of 2D foams resembling images of conformal crystals as observed in phytotaxy [1] or in diluted suspensions of ferro-fluid balls [2,3]. In the ferro-fluid foams [4], the conformal distortion is due to gravity. [5] used special Hele–Shaw cells with an upper plate either tilted or curved. They attributed the conformal transformation behavior to volume conservation; as the thickness $h$ varies, the 2D area of the quasi-cylindrical bubbles varies in inverse proportion. In [6], the two hemispherical walls remain parallel.

The understanding of the physical origin of this conformal behavior is not entirely clear. Below, we will prove that conformal invariance is a genuine property of the Plateau equilibrium equations at the vertices. On the other hand, the only functions conformally mapping circles to circles are the homographies, or Moebius transformations, composed of 2D similarities and inversions [7,8]. At equilibrium, since the gas pressure is constant in the bubbles, Laplace’s law implies that each film has constant curvature, and hence is a circular arc in 2D. So any conformal mapping other than homographies seems to contradict Laplace’s law. We propose a solution to this paradox by lifting the problem back to 3D space. Films suspended between non-parallel plates bend, with a vertical curvature contributing to the pressure balance.

Before we analyse specific cases, we formulate the equilibrium equations for the 2D foam traced out when a 3D foam contacts a solid surface. This surface may be curved, but it must be smooth and clean to prevent film pinning by irregularities and ensure relaxation to equilibrium. Otherwise, these equations hold in quite general set-ups, not only for the standard flat and parallel Hele–Shaw cell, as long as the incidence of the films or interfaces is normal to the wall.

Previous studies [13,14] treated foams in curved surfaces. However, the perspective was different. The focus was mainly on coarsening due to diffusion for foams confined in surfaces, whence intrinsically 2D. The paradoxes raised by analysing...
conformal images in terms of purely 2D physics led us to re-
consider the problem as a contact with a boundary, involving
both 2D and 3D features.

The outline of the article is as follows: After some basics
on foams equilibrium in Section 2, the contacts and the equi-
librium equations at the boundary are described in Section 3
for the case of normal incidence. Section 4 establishes con-
formal invariance for nodal equilibrium. Section 5 addresses
the question of conformal mappings for the edges and solves
a specific example to lowest order in non-parallellism. Section 6
concludes.

2. Foam equilibrium

Soap films obey the Young–Laplace law:
\[ \Delta P + 2\gamma H = 0, \]  
(1)
where \( \Delta P = P_1 - P_2 \) is the pressure difference across the
film, \( \gamma \) is the surface tension and \( H \) is the mean curvature of
the film.

Plateau showed that the films and cells meet three by three at
edges and satisfy \([9,10]\):
\[ \sum_{i=1}^{3} \gamma_i \mathbf{b}_i = 0, \]  
(2)
\[ \sum_{i=1}^{3} \gamma_i H_i = 0, \]  
(3)
where Eq. (2) asserts that the sum of the forces on any edge
is zero; the force from the film \( \mathcal{F}_i \) points in the direction
given by the unit vector \( \mathbf{b}_i \) tangent to the film and normal to
the junction. In general, the surface tension \( \gamma_i \) may differ on
different films. A \( \pi/2 \) rotation around the axis tangent to the
Plateau border converts the first equilibrium Eq. (2) to
\[ \sum_{i=1}^{3} \gamma_i \mathbf{n}_i = 0, \]  
(4)
where \( \mathbf{n}_i \) are the normal vectors to the films.

Eq. (3) states that, by (1), the sum of the pressure drops
around an edge, as on any closed path, is zero. A sign con-
vention in (3) gives the sense of the circulation; the pressure
difference in going from bubble 1 to bubble 2 is defined as
\( P_2 - P_1 \). Via (1), this definition locally fixes the orientation
of the three films around any given edge or Plateau border.

3. Contact with a boundary

Assume that the foam is in contact with a solid wall \( \mathcal{S} \),
smooth and clean, so that the films can freely slide along the
surface.

An interface \( i \) has a contact with the wall only in non-
wetting conditions; then the contact angle \( \theta \) satisfies Young’s
law:
\[ \gamma_i \cos \theta + \gamma_{S1} - \gamma_{S2} = 0, \]  
(5)
where 1 and 2 label the two fluids (gas in normal foams) on
both sides of the film.

In the case of a film – a thin double interface – the two
bubbles contain the same gas so that \( \gamma_{S1} = \gamma_{S2} \) and \( \cos \theta = 0 \).
This condition of normal incidence is also valid when the
soap solution wets the solid surface, provided the wetting
films on both sides have the same surface tension.

Our purpose, next, is to see what happens if the glass
surface is not planar.

3.1. Equilibrium conditions at a boundary

The contacts of the films in a 3D foam with a solid bound-
dary \( \mathcal{S} \) form a 2D foam on \( \mathcal{S} \).

Consider a Plateau border \( \psi \) where three films \( \mathcal{F}_i, i = 1, 2, 3 \) meet. These films and the border contact the smooth
surface \( \mathcal{S} \) through a triple of lines \( \phi_i \) and a vertex \( p \) of
the 2D foam in \( \mathcal{S} \) (Fig. 1). Let us parameterise the curves by arc
length \( s \) with \( s=0 \) at \( p \). For any curve \( \phi, \tau = \phi = (d/ds) \) is the
unit vector tangent to \( \phi \).

Assume normal incidence on \( \mathcal{S} \), that is, \( |\theta| = \pi/2 \). Then,
at \( p \), common to \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \) and \( \mathcal{S} \), the following hold:
\[ \sum_{i=1}^{3} \gamma_i \mathbf{t}_i = 0, \]  
(6)
\[ \sum_{i=1}^{3} \gamma_i k_{\mathcal{S}}(\phi_i, \mathcal{S}) = 0, \]  
(7)
where \( k_{\mathcal{S}}(\phi, \mathcal{S}) \) is the geodesic curvature of \( \phi \) in \( \mathcal{S} \) \([11]\).

When all the film tensions are equal, \( \gamma_i = \gamma \), the triangles
representing the vector sums in (2) and (6) are equilateral and
the Plateau angles are \( 2\pi/3 \).

Because of normal incidence, the tangent \( \mathbf{t}_i \) coincides with
the co-normal \( \mathbf{b}_i \) at \( p \). So (6) follows immediately from (2).
See ref. \([12]\) for a full derivation of (7).

4. Conformal invariance of bidimensional foams

We may separate the equilibrium equations into two
groups.

First, Laplace’s equation (1) constrains the shape of the
films: It involves the mean curvature \( H \) which we cannot, in
general, determine by looking only at the contact lines with
the solid surface \( \mathcal{S} \). One notable exception is the Hele–Shaw
cell: when the foam is squeezed between parallel plates, flat
or curved, but close to each other, the cells are cylinders, or
conical sections, with zero vertical curvature. In other words,
the films only curve in the direction parallel to the surface
so that the geodesic curvature is proportional to the pressure.
difference:
\[ P_2 - P_1 = 2\pi \alpha = \kappa g. \]

The second group comprises the two Plateau equations (6) and (7) which specify the equilibrium conditions at the nodes of the 2D foam. These equations somehow decouple from the full 3D problem. If we reduce the constraints to these nodal equations, then they are conformally invariant; the image of any solution, after applying a conformal map \( f \), is still a solution, as we show in this section. As a further generalisation, the property turns out to be true for any vertex degree \( d \) (or coordination number). Plateau says that any solution, after applying a conformal map \( f \), satisfies

\[ \Delta = 2 \]

Indeed, the Cauchy–Riemann condition,

\[ \partial_x f = \frac{1}{2} (\partial_1 - i \partial_2) f + i \frac{1}{2} f = \partial_1 f^1 + i \partial_2 f^2, \]

where \( f^1 = \partial_1 f = \partial_1 f^1 \) and \( f^2 = \partial_2 f = -\partial_1 f^1 \).

The scalar product in \( \mathbb{R}^2 \) translates as \( x \cdot y = \text{Re}(x^* y) \) for any complex \( x, y \).

### 4.2. Conformal maps

The twice differentiable map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), defined as

\[ x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto \tilde{\phi} = f^{1}(x^1, x^2) \]

is conformal if and only if its Jacobian matrix \( J^f \) satisfies

\[ J^f = \lambda^2 I \]

for some positive \( \lambda = \lambda(x) \).

2D conformal maps are equivalent to complex holomorphic functions through the identification \( \mathbb{R}^2 \leftrightarrow \mathbb{C} \),

\[ x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto z = x^1 + ix^2. \]

Indeed, the Cauchy–Riemann condition,

\[ \partial_x f = \frac{1}{2} (\partial_1 - i \partial_2) f + i \frac{1}{2} f = \partial_1 f^1 + i \partial_2 f^2, \]

is equivalent to (13). Then

\[ f(z) = \frac{1}{2} (\partial_1 - i \partial_2)f = f^1 + i f^2, \]

where \( f^1 = \partial_1 f = \partial_1 f^1 \) and \( f^2 = \partial_2 f = -\partial_1 f^1 \).

The scalar product in \( \mathbb{R}^2 \) translates as \( x \cdot y = \text{Re}(x^* y) \) for any complex \( x, y \).

### 4.3. Paths under conformal maps

Consider a path \( \phi \) and its image \( \tilde{\phi} \) under a conformal mapping \( f : \mathbb{C} \to \mathbb{C} \) such that \( \tilde{\phi} = f(\phi) \).

The transformation rules are:

\[ \tilde{\phi} = f(\phi). \]
\( \hat{\psi} = f' \psi + f''(\psi)^2, \) \hspace{1cm} (17) \\
\hat{t} = \frac{f'}{|f'|} \tau, \quad \hat{v} = \frac{f'}{|f'|} v, \quad \hat{n} = \frac{f'}{|f'|} n, \hspace{1cm} (18) \\
\hat{k} = \left( \frac{i \phi \psi}{|\psi|^2} + \frac{f'}{|f'|} \right) \tau - \frac{f''(\psi)^2}{|f'|^2} \tau, \hspace{1cm} (19)

where \( f' \equiv f'(\psi(t)) \), etc.

4.4. Conformal invariance of nodal equilibrium

Let \( X = (V, E, \gamma) \) be a graph in the plane with vertex set \( V \), edges \( E \) (all twice differentiable curves) and line tensions \( \gamma : E \to \mathbb{R} \).

For any conformal map \( f \) of the plane, the Plateau conditions ((6) and (7)) are satisfied at all vertices of \( X = f(X) \) if and only if they are satisfied by \( X \).

To check this proposition, let \( x \) be a vertex of the graph and \( \hat{x} \) its image, \( x \) is a point common to all the edges \( \phi_j \) incident on \( x \), \( j = 1, \ldots, d \).

The equivalence for (6) is an immediate consequence of conformality: let \( e^{\alpha z} = f'(x)/f'(x) \); then, by (18), \( \sum |\gamma_j \tau_j| = e^{\alpha} \sum |\gamma_j \tau_j| \), showing that condition (6) for \( \hat{X} \) holds if and only if it holds for \( x \).

To show that (7) for \( x \) is equivalent to (7) for \( \hat{x} \), we may use (19):

\[
\sum_{j=1}^{d} |\gamma_j \hat{\tau}_j| = \frac{1}{|f'|} \left( \sum_{j=1}^{d} |\gamma_j \tau_j| + \text{Re} \left( \frac{f'}{f'} \sum_{j=1}^{d} |\gamma_j \tau_j| \right) \right),
\]

where everything is evaluated at \( x \). Using (6), we see that the sum of the curvatures around \( \hat{x} \) vanishes if and only if the sum around \( x \) does.

Drenckhan et al. [5] gave an alternative argument, based on homographies and Taylor expansion to second order.

5. Mapping the edges: the example of the log

In the previous sections, we have seen that the equilibrium conditions at the nodes are invariant under conformal maps. Here, we consider the effect of a specific conformal map on the edges. The logarithmic map is one of the examples experimentally demonstrated in [5]:

\[ F(z) = \frac{1}{2\pi i} \ln(izc), \] \hspace{1cm} (21)

where \( a \) is a real parameter. This mapping is also relevant to gravity arches observed in ferro-fluid suspensions under gravity [2,3] and to ferro-fluid 2D foams [4]. The logarithm is the only conformal map translationally invariant [2,3,15].

Under mappings such as the logarithm, the image of a circular arc is not a circular arc. We first evaluate the curvature to see how close the curve is to a circle. Then we will argue that, in the circumstances under study, the edges do not need to be circular to satisfy the equilibrium conditions. Under gravity, as in [4], we must add the gravity field to the edge equations (1) or (8). When the foam lies between two non-parallel plates, curvature in the third dimension modifies the equations with respect to the usual 2D formulae, as explained below (Section 5.2).

5.1. Log map of an edge

The inverse \( f(z) = F^{-1}(z) = \exp(iazc)/ia \) maps a foam \( X \) in a non-parallel chamber to a reference one, \( \hat{X} \). The function \( f \) satisfies \( f' = a\psi \). Inverting Eq. (19) and applying it to \( f \) gives the curvature vector:

\[ k = \hat{k} f' + \text{Re} \left( \frac{f''}{f'} \right) = a (\hat{\psi} - \text{Re}(\tau)). \] \hspace{1cm} (22)

In (22), the first term on the right is the effect of the local scale change by \( |f'| \); according to the second term, edges nearly parallel to the imaginary axis are almost undeformed, up to similarities, whereas those parallel to the real axis bend. See Fig. 3.

5.2. Foam between non-parallel plates

Suppose that the foam is sandwiched between two plates, as in a conventional Hele-Shaw cell, except that the upper plate is not parallel to the bottom one, as in [5]. We may describe the upper plate face wet by the foam by a height function \( h(x^1, x^2) \) defined in the plane at the bottom of the cell. We assume that both \( h \) and its slope \( \nabla h \) are small. The 2D vector \( \nabla h \) locally indicates the direction of maximal slope. The slope in this direction is the local dihedral angle (which is small). Because the film meets both top and bottom faces orthogonally, it must curve: its sectional, or normal, curvature in the vertical direction is \( k_v = -\text{Re} \cdot \nabla h / h \) (Fig. 4). At the same degree of approximation, we can approximate the
mean curvature $H$ by adding the horizontal sectional curvature $k \equiv k_\perp(\psi, S)$ of the curve $\psi = f \cap S$, $S$ being the upper or lower boundary. Then, Laplace’s law says

$$P_1 - P_2 = 2\pi H = \gamma (k + k_\parallel) \equiv \gamma (k - \frac{n.F}{h}) \tag{23}.$$  

Thus, in equilibrium, with pressure difference constant along the film, the geodesic curvature differs from a constant by the last term in (23), representing contributions from curvature in the third direction.

Example

Consider the exponential profile $h(z) = h_0 \exp(\gamma \cdot z)$, where $\gamma$ is a fixed small vector in the bottom plane. In this case $k_\parallel \equiv -\gamma \cdot n$.

The exponential profile gives a contribution matching that of the log map if we take $g = -\gamma$ and $a_\parallel = 0$. Indeed, as $n.F / h = \pi$ and $k_\parallel \equiv -\gamma \cdot n = \Re(g \tau)$. Comparing (22) and (23), we get $\alpha \tau = \frac{1}{\gamma}$ and $|\frac{n.F}{h}| = (P_1 - P_2)^2 / \gamma$. If the reference flat foam is at constant pressure ($P = 0$), e.g. a hexagonal foam [5], we conclude that the conformal image between non-parallel plates is also at constant pressure; a vertical contribution, invisible in the projection, compensates the curvature apparent in the 2D projection.

6. Conclusion

When a foam is in perpendicular contact with a smooth solid surface, the 2D foam formed by the contacts satisfies the 2D Plateau equilibrium conditions at the nodes. The case of oblique, non-normal incidence is more involved and no simple separation appears to occur of the 2D from the 3D problem.

The equilibrium equations for the nodes, (6) and (7), are conformally invariant, as we have proved in Section 4.

While films are circular arcs in standard 2D foams, they may deform, either under external fields such as gravity or in the third direction.

Fig. 4. A film joining two non-parallel plates.

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In fact, the pressure difference is the sum of two curvatures. Along a contact curve, these may be the geodesic (longitudinal) and normal curvatures, which can both vary along the curve even if the sum, $2H$, is constant. For a chamber bounded by two smooth, non-parallel plates, with exponential profile, the 2D conformal image has non-constant geodesic curvatures $k_\parallel$ but Laplace’s law nevertheless holds, provided we consider the films as curved surfaces in space.

However, comparison with the available experiments is not straightforward. The transformations of [5] were done at constant bubble volume, implying $h \propto |f|^2$ if the bubbles are small enough, whereas our theory, based on film equilibria, yields $h \propto |f|^1$, but at constant pressure. Except for the trivial case where the plates are parallel, these relations differ.

Is our assumption of nearly constant vertical curvature too crude? Another possibility is that the precise mappings describing the geometry at long and short length scales might differ. One mapping, e.g. obeying the constant volume or constant pressure condition, would describe the global order, the bubbles arrangement, in a way similar to that for magnetic suspensions [2]; whereas the local geometry, at the scale of individual films or bubbles, would agree with another mapping following from the local equilibrium conditions as we have shown. This point is further discussed in [12].

Combining these two length scales raises challenging questions. Similar experiments carried out at constant pressure rather than constant volume would be interesting.

The integral of the geodesic curvature is one of the contributions to the Gauss-Bonnet theorem. For intrinsically 2D foams, this term sums up to a topological quantity in the evolution equation [16]. Another contribution is the integrated Gaussian curvature [13,14]. There were also proposals to find 3D analogues of von Neumann’s law [13,17]. How the foam coarsens near a solid, impermeable, wall is yet an interesting question.

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References


