DEFORMATIONS OF PERIODIC MINIMAL SURFACES

C. OGUEY
LPTM, Université de Cergy-Pontoise
2 av. A. Chauvin
F - 95302 Cergy-Pontoise

Abstract. [1] Periodic minimal surfaces model a large variety of mesophases and crystals of films in ternary liquid systems typically composed of water, soap and oil. First I describe the Bonnet transformation with emphasis on the crystallographic aspects which are specific to periodic minimal surfaces. Then I consider other transformations which are non isometric but preserve minimality.

1. Introduction

Mesophases, or sponge phases, are liquid crystals where the ordering (lamellar, hexagonal, cubic etc.) is due to the presence, in the core of the phase, of a film made of either water (direct) or oil (reverse) sandwiched between two layers of surfactant molecules with appropriate orientation. The space left is filled, in majority, by the third component [1, 3, 21]. In binary systems with similar structures, the film is just a bilayer of amphiphiles. I leave aside the micellar phases which generally occur at extreme concentrations (high or low water proportion) (see J. Seddon in this volume). At intermediate concentrations of the mixture, the space is partitioned by a liquid film. For the morphology of those films, the standard models are minimal surfaces. This is a feature common with bubbles or foams.

There are however differences. 1) The length scale is typically smaller; in the cubic phases, for example, the lattice parameter is of the order of a few nanometers. 2) The relaxation times are reasonably short allowing thermodynamic equilibrium to set in. In particular, the pressure is constant. 3) Except for the micellar phases, the structure is not a packing of closed cells. Depending on the values of temperature and concentrations, the phase

[1]Chap. XXVII of *Foams and Emulsions* (NATO E, 354), J.F. Sadoc and N. Rivier eds, Kluwer (1999)

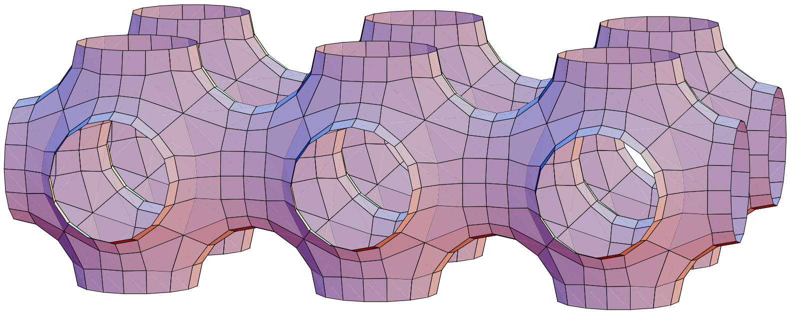

Figure 1. Example of a triply periodic minimal surface: the P surface.

may consist of tubes (hexagonal), lamellae or intertwined labyrinths as in the cubic phases. 4) In the cubic phases, or their variants with similar geometries but lower symmetries, there are no singularities, no edges, no Plateau borders etc. (fig. 1). Those are the ones which I will mostly consider, at least when looking at non local features like topological changes.

When the pressure is equal on both sides, soap bubbles take the shape of minimal surfaces. Locally minimizing the area with fixed boundaries (such as wire frames) yields the equation $H = 0$ where $H = (k_1 + k_2)/2$ is the mean curvature, arithmetic mean of the principal curvatures k_1 and k_2. In fact, after Laplace, the mean curvature H is precisely equal to the pressure difference in situations close to equilibrium. So the surface energy is simply

$$E = \sigma A = \sigma \int_M dA, \qquad (1)$$

where M is the surface embedded in space and σ is surface tension.

At a lower scale, when we want to describe the interfacial films in the mesophases, Helfrich's form for the energy is a good approximation [7] :

$$E = \int_M (\kappa H^2 + \bar{\kappa} K) dA + LR. \qquad (2)$$

The factors $\kappa, \bar{\kappa}$ are elastic constants; LR denotes longer range interactions such as those responsible for the spacing between lamellae [14].

In mathematics, surfaces are often viewed as abstract objects of dimension 2 with intrinsic metrics and topology [22, 23]. This allows to describe, with more precision, surfaces such as the Klein bottle or the projective plane which cannot be embedded in our familiar Euclidean space R^3. The mean curvature H, however, hence minimality, is not an intrinsic property; it relates to the way the surface is immersed in the Euclidean space

$E = R^3$. Therefore one often speaks of minimal immersions or embedding rather than minimal surfaces. On the other hand, the Gaussian curvature, equal to the product of the principal curvatures $K = k_1 * k_2$, can be shown to be intrinsic. The mathematical literature on minimal surfaces is abundant. I can only mention textbooks [2, 11, 19, 17] and a selection of articles [13, 9, 10, 15].

For modeling physical systems, the surface has to be taken as it is embedded in space, this is clear. It is of physical relevance, nevertheless, to know of what type the most important terms in the energy are. By important I mean the ones determining the shape, the flexibility and low excitations of the film. I will not solve that problem here but get a few hints.

2. Immersion in the complex space C^3

The equation $H = 0$ is a non-linear second order partial differential equation for the coordinates $\mathbf{x} = (x_1, x_2, x_3)$ which is solved by the Weierstrass-Enneper formula (WE)

$$z \to \mathbf{x}(z) = \mathrm{Re}\,(a\,\xi(z))\,, \tag{3}$$

$$\xi(z) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \int_{z_0}^{z} \begin{pmatrix} 1 - w^2 \\ i(1 + w^2) \\ 2w \end{pmatrix} R(w)\,dw. \tag{4}$$

Re denotes the real part. This formula gives the Cartesian coordinates of a point of the surface $\mathbf{x}(z)$ as a function of the complex parameter $z = u + iv$ (equivalent to two real ones, as necessary for two dimensional objects). It involves an analytic function R (so that $R\,dw$ is a meromorphic differential). The integration runs along a path from z_0 (fixed) to z (running) in the complex plane C. By Cauchy's theorem, the value of the integral is independent of the path for all paths homotopic in the domain of definition of R, which is a Riemann surface.

This formula is largely explained in the textbooks. Let me just mention two points, setting $\eta = \xi' = \partial_z \xi$:

- $\eta \cdot \eta \equiv \eta_1^2 + \eta_2^2 + \eta_3^2 = 0$. That identity reflects the fact that the coordinates $z = u + iv$ are conformal ($z \to \mathbf{x}(z)$ preserves angles).
- $\partial_{z*}\eta = 0$. This is the minimality condition which turns out to be equivalent to the Cauchy-Riemann condition for analyticity.

The module of the complex number a fixes the length-scale whereas its phase provides, upon variation, different minimal surfaces. Changing the phase $\theta = \arg(a)$ in (3) is known as a Bonnet transformation. Two surfaces related by such a transformation are in general different (they cannot be brought to coincidence by any Euclidean motion) but they are locally isometric as surfaces.

When the parameter z is restricted to vary within a suitable domain, free from singularities of R, the WE formula provides an analytic 1-1 map between the domain and a patch of the surface M. However such a restriction is not necessary. By analytic continuation, the integration can be carried along any path avoiding the singularities (poles and branch points in the cases under study). Of course non homotopic paths to some value z of the parameter may give different values for the coordinates $\mathbf{x}(z)$, and so, different points in R^3. The vector joining two such points is a period, in mathematical parlance, that is, a vector of the translation lattice of the surface. This is the way periodic surfaces can be expressed in a single formula.

The most well known triply periodic surfaces are Schwarz' P (primitive), D (diamond or face centered F) and G (gyroid, body centered) [18, 20]. They all three have cubic symmetry and their WE differential is

$$R(w) = (1 + 14w^4 + w^8)^{-1/2}. \tag{5}$$

Albeit different regarding their geometrical aspect (different space groups, lattice parameters etc..), those three surfaces only differ, in formula (3-4), by the Bonnet angle θ which has value $0°$, $90°$ and $51,985..°$ respectively.

To study the Bonnet transformation, the most natural thing to do is to investigate the surface in C^3 which is a lift (covering) of all the (three in this case) members of a Bonnet family. What the WE formula (4) provides is, first of all, an immersion of the surface into $C^3 \simeq R^6$: $z \to \xi(z)$. The real surface is then obtained by linear orthogonal projection down to the physical space: $\mathbf{x} = \mathrm{Re}(a\xi)$. Note that, locally, this projection produces no singularity on the surface since the metric tensors of the complex surface ξ and of the real projected one $\mathbf{x}$ are related by a constant factor 1/2.

The translation symmetries of the real surface are projections of the translations of the complex one. Those translations are obtained by contour integration along closed loops; they are thus tightly related to the topology of the underlying Riemann surface. For example, the three surfaces P, D , G of genus $g = 3$ per unit cell, have a common covering, which I will call PDG, and which is a periodic analytic surface embedded in C^3. On the Riemann surface $R(w)^{-2} = 1 + 14\,w^4 + w^8$ the map (4) has six independent (over the reals) complex periods generating a 6 dimensional lattice Λ, the symmetry lattice of the complex surface PDG. A possible set of generators for Λ is (see [18] for details)

$$(t_1, t_2, t_3, \tau_1, \tau_2, \tau_3) = \begin{pmatrix} r & -is & is & 0 & r-is & -r-is \\ is & r & -is & -r-is & 0 & r-is \\ -is & is & r & r-is & -r-is & 0 \end{pmatrix},$$

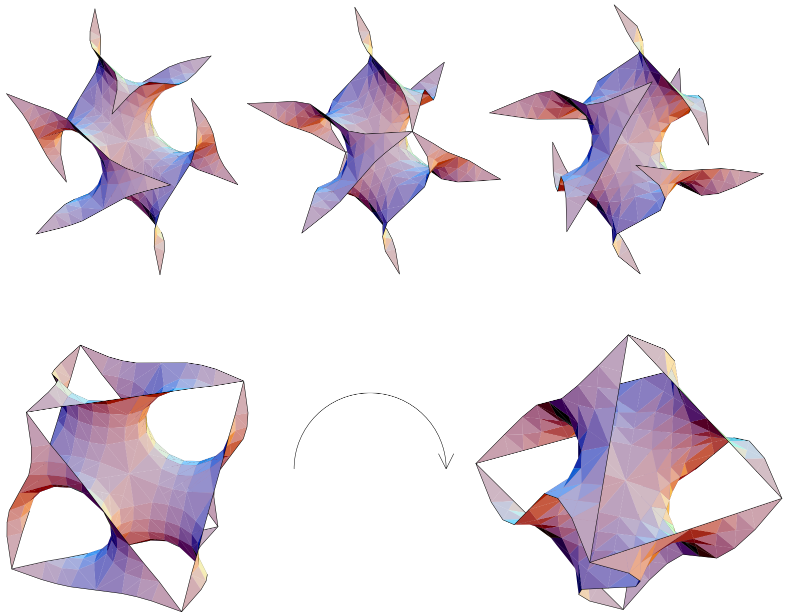

Figure 2. The Bonnet transformation acting on a fundamental domain of the surface. The angle has successive values 0 (P), 30, 52 (G), 70, 90° (D).

where $r = 2.156..$ and $s = 1.686..$ are the cubic parameters of the P and F surfaces respectively. The global scaling factor is, of course, arbitrary; here it is set to $|a| = 1$. The ratio of the two parameters r and s is, however, not arbitrary because it has to fulfill the isometry condition : patches mapped onto each other by the Bonnet transformation must have the same area. This fixes r/s to be 1.279...

3. The Bonnet transformation

As we have just seen, the Bonnet transformation is an isometry on bounded patches (distances and angles within the surface are preserved). Now, in mesophases, the interface or the film is believed to form an infinite bicontinuous surface. So it seems natural to look at the transformation as acting globally, or at least on larger patches.

Consider the example of the P surface which is typical in these respects.

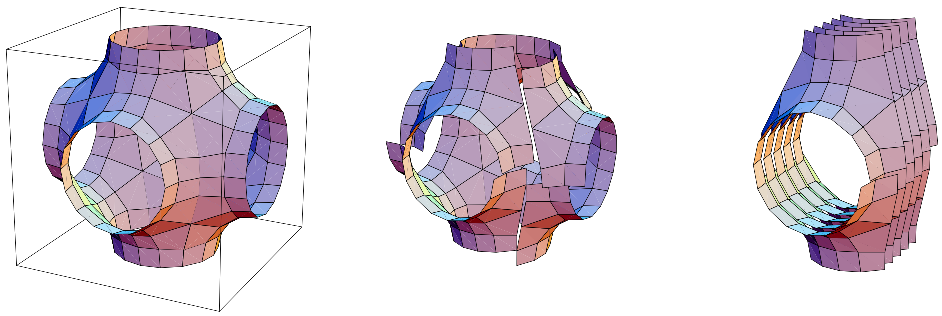

Figure 3. Fundamental piece of the P surface at Bonnet angle $\theta = 0$ (left), and at $\theta \simeq 3^\circ$ (center and right) with cuts or self-intersections.

Starting from the fundamental piece shown on fig. 3 (left), corresponding to $\theta = 0$, let us raise θ to a tiny non zero value. Then two requirements are possible. Either we roughly fix the total area of that piece (this would correspond to conservation of the film material, which does not need to be exactly conserved because of the liquid reservoir) and the piece tears itself (fig. 3 center). Or we impose bicontinuity (no boundaries nor tearing), but then the surface displays sequences of helicoidal convolutions with many self intersections, the whole surface ending in densely filling space like a bended millefeuille (fig. 3 right). None of those features sounds physically plausible.

The Bonnet transformation is, however, a suitable way to design other examples of minimal surfaces, provided it is kept in mind that physically reasonable candidates occur only for a finite subset of values of the Bonnet angle: rational $(s/r)\tan\theta$ with small numerator and denominator.

Another issue of the Bonnet transformation is that it indicates the relevance of the LR term in (2). Indeed an energy function depending only on the local metric, as the first term in (2), does not depend on θ, therefore it has the same value on all the members of a Bonnet family. So if, at some thermodynamic coordinates, the system adopts a definite structure without degeneracy, choosing one (for example the G) rather than another (the P or D) Bonnet equivalent conformation, it is necessarily due to non-local interactions and, eventually, entropic contributions.

4. Non isometric transformations

Probably closer to real physical processes undergone by the film under stress or low energy excitations, the continuous non isometric deformations form a large class, even with suitable constraints (steric barriers, bounds on

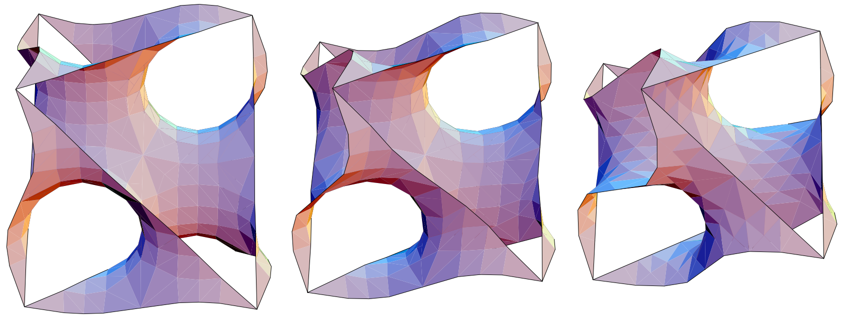

Figure 4. Continuous rhombohedral distortion of the P (left) and the D (right) surfaces.

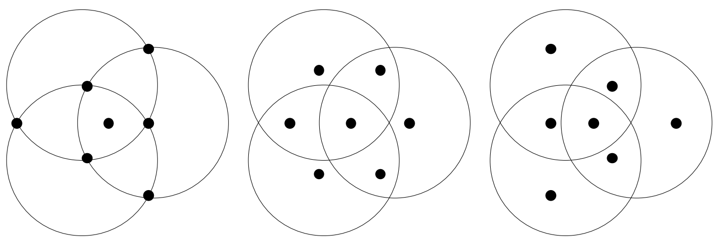

Figure 5. Movement of the branch points corresponding to fig. 4. In this representation, zero and infinity are branch points and stay fixed. The other ones move radially.

curvatures etc.). A thermodynamic treatment would then be appropriate [16].

Let me focus on one subclass which will be of interest for systems with large elastic constant κ in (2) (in the elastic as well as non-linear regime). For each minimal surface, there is a set of continuous deformations which are not isometries but which preserve minimality: the deformed surfaces still satisfy $H = 0$ everywhere. A number of examples have been studied by the Australo-Swedish group [4, 5, 6, 12]. The salient features of these deformations are, first, that they are bounded in some directions, reaching instability beyond a certain threshold [8], second, that, given a minimal surface with finite topology (finite genus for non periodic MS in R^3, finite genus per unit cell for periodic ones), the set of continuous deformations preserving $H \equiv 0$ is finite dimensional. Considering such a family of minimal surfaces or immersions, it may happen that the Bonnet conjugates fall in the same family. In that case, the family is said to be self adjoint.

A well known example is the rPD family. Uniaxial compression-dilation along a 3-fold axis induces a continuous transition from the P surface to the D and vice versa. The intermediate surfaces are all minimal with lower symmetries (fig. 4). The G is not reached this way because the path followed here is different from the Bonnet route even though the end points (P and D) are the same.

To study those deformations, or classes, of minimal surfaces, the WE representation is well suited. The parameters controlling the morphological changes are those entering the WE function [13, 5]. For example, triply periodic MS with genus 3 per unit cell form a family whose dimension is, a priori, not larger than 18. They are characterized by a WE form $R(w) = (a_0 + a_1 w + ... + a_8 w^8)^{-1/2}$ involving a polynomial of degree 8. The parameters can be taken to be the complex coefficients of that polynomial.

Among these degrees of freedom, 4 are trivial (3 corresponding to global rotations, 1 to change of length scale). A number of others will lead to non embedded surfaces (with self-intersections and without minimal bound on the distance between sheets). In order to get a regular embedded surface with a true 3D lattice as part of the symmetry group, 3 basis vectors need to cancel in the projection from 6D (where the complex PDG surface is). Algebraically, this amounts to 9 scalar constraints leaving us with 5 degrees of freedom for the set of genuine deformations of regularly embedded, triply periodic, minimal surfaces with $g = 3$. (This is assuming that the 9 constraints are independent; otherwise, this set might be larger). That number 5 is also the dimension of the space of linear volume preserving deformations in elasticity (symmetric 3×3 tensors with vanishing trace).

To my knowledge, a systematic investigation of the deformed MS is still lacking, even for the case $g = 3$. Meeks [13] has a set V of dim. 5 which includes many examples of interest but it is not complete because it does not contain G. This is still under investigation.

Acknowledgments. I wish to thank S. Hyde, N. Rivier, J-F. Sadoc and J. Seddon for discussions. Special thanks to the organizers of the school.

References

1. Andersson, S., Hyde, S.T., Larsson, K., Lidin, S. (1988) Minimal surfaces and structures: from inorganic and metal crystals to cell membranes and biopolymers *Chem. Rev.* **88**, 221-242.
2. Dierkes, U., Hildebrandt, S., Kuster, A., Wohlrab, O. (1991) *Minimal Surfaces I & II* Springer, Grundl. der math. Wiss. vol. 295 & 296.
3. Dubois-Violette, E., Pansu, B. eds (1990) *Internationnal Workshop on Geometry and Interfaces , J. Phys. Colloq. France* **51** C7.
4. Fogden, A. (1992) A systematic method for parametrising periodic minimal surfaces: the F-RD surface *J. Phys. I France* **2** 233-239.
5. Fogden, A., Hyde, S. T. (1992) Parametrization of triply periodic minimal surfaces *Acta Cryst. A* **48** 442-451 and 575-591.
6. Fogden, A., Heaberlein, M., Lidin, S. (1992) Generalization of the gyroid surface *J. Phys. I France* **3** 2371-2385.

7. Helfrich, W. (1973) *Z. Naturforsh.* **28C**
8. Hoffman, D. A. Some basic facts, old and new, about triply periodic embedded minimal surfaces , in [3].
9. Hoffman, D. A., Meeks, W. H. III (1990) Minimal surfaces based on the catenoid, *American Math. Monthly* **97**.
10. Hoffman, D. A., Osserman, R. (1985) The Gauss map of surfaces in R^3 and R^4 *Proc. Lond. Math. Soc.* **50** 27-56.
11. Lawson, H. B. (1971) *Lectures on Minimal Submanifolds* Berkeley, Publish or Perish.
12. Lidin, S., Hyde, S. T. (1987) A construction algorithm for minimal surfaces *J. Phys. France* **48** 1585-1590.
13. Meeks, W. H. III (1990) The theory of triply periodic minimal surfaces, *Indiana U. Math. J.* **39** 877-935.
14. Mosseri, R., Sadoc, J. F. (1997) *Frustration Géométrique* Alea Saclay, Eyrolles. English version: (1997) *Geometric Frustration* Cambridge.
15. Nagano, T., Smyth, B. (1980) Periodic minimal surfaces and Weyl groups, *Acta Math.* **145** 1-27.
16. Nelson, D. R. *et al.* edts (1989) *Mechanics of Membranes and Surfaces* World Scientific.
17. Nitsche, J. C. (1989) *Lectures on Minimal Surfaces* Cambridge.
18. Oguey, C., Sadoc, J. F. (1993) Crystallographic aspects of the Bonnet transformation for periodic minimal surfaces, *J. Phys. I France* **3** 839-854.
19. Osserman, R. (1986) *A Survey of Minimal Surfaces* Dover.
20. Sadoc, J. F., Charvolin, J. (1989) Infinite periodic minimal surfaces and their crystallography in the hyperbolic plane *Acta Cryst. A* **45** 10-20.
21. Seddon, J. (1996) Lyotropic phase behaviour of biological amphiphiles, *Ber. Busenges. Phys. Chem.* **100** 380-393.
22. Spivak, M. (1979) *A Comprehensive Introduction to Differential Geometry* Houston, Publish or perish.
23. Willmore, T. J. (1993) *Riemanian Geometry* Oxford UP.